

ON RANK OF THE JOIN OF TWO SUBGROUPS IN A FREE GROUP

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ABSTRACT. Let H, K be two finitely generated subgroups of a free group, let $\langle H, K \rangle$ denote the subgroup generated by H, K , called the join of H, K , and let neither of H, K have finite index in $\langle H, K \rangle$. We prove the existence of an epimorphism $\zeta : \langle H, K \rangle \rightarrow F_2$, where F_2 is a free group of rank 2, such that the restriction of ζ on both H and K is injective and the restriction $\zeta_0 : H \cap K \rightarrow \zeta(H) \cap \zeta(K)$ of ζ on $H \cap K$ to $\zeta(H) \cap \zeta(K)$ is surjective. This is obtained as a corollary of an analogous result on rank of the generalized join of two finitely generated subgroups in a free group.

1. INTRODUCTION

It is a recurring theme in group theory to embed a countable group into a 2-generated group often with some additional properties, see [1], [3], [4], [7], [11]. In this article we will look at two finitely generated subgroups of a free group from a similar perspective.

Let H, K be two finitely generated subgroups of a free group F . Let $S(H, K)$ denote a set of representatives of those double cosets HgK , $g \in F$, for which the intersection $H \cap gKg^{-1}$ is nontrivial.

According to Walter Neumann [9], the set $S(H, K)$ is finite and a formal disjoint union $\bigvee_{s \in S(H, K)} H \cap sKs^{-1}$ of subgroups $H \cap sKs^{-1}$, $s \in S(H, K)$, could be considered as a generalized intersection of H and K . Let $r(F)$ denote the rank of a free group F and let $\bar{r}(F) := \max(r(F) - 1, 0)$ denote the reduced rank of F .

Theorem 1.1. *Suppose that H, K are two finitely generated subgroups of a free group F , $L = \langle H, K, S(H, K) \rangle$ denotes the subgroup generated by $H, K, S(H, K)$, and neither H nor K has finite index in L . Then there is an epimorphism $\eta : L \rightarrow F_2$, where F_2 is a free group of rank 2, such that the restriction of η on H and K is injective, a set $S(\eta(H), \eta(K))$ for subgroups $\eta(H), \eta(K)$ of F_2 can be taken to be $\eta(S(H, K))$, and, for every $s \in S(H, K)$, the restriction*

$$\eta_s : H \cap sKs^{-1} \rightarrow \eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$$

of η on $H \cap sKs^{-1}$ to $\eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$ is surjective.

Informally, we can say that, when given a *generalized join*

$$L = \langle H, K, S(H, K) \rangle$$

of two finitely generated subgroups H, K of a free group F such that neither of H, K has finite index in L , it is always possible to assume that $r(L) = 2$, i.e., the subgroup $\langle H, K, S(H, K) \rangle$ is 2-generated. Since a free group of an arbitrary finite

2010 *Mathematics Subject Classification.* Primary 20E05, 20E07, 20F65, 57M07.
Supported in part by the NSF under grant DMS 09-01782.

rank is isomorphic to a subgroup of a free group of rank 2, it is easy to obtain the equality $r(F) = 2$ and our goal here will be to attain the equality $r(L) = 2$.

We mention an easy corollary of Theorem 1.1 for a “pure” intersection case.

Corollary 1.2. *Suppose that H, K are two finitely generated subgroups of a free group F , $\langle H, K \rangle$ denotes the subgroup generated by H, K , and neither H nor K has finite index in their join $\langle H, K \rangle$. Then there is an epimorphism $\zeta : \langle H, K \rangle \rightarrow F_2$, where F_2 is a free group of rank 2, such that the restriction of ζ on H and K is injective and the restriction $\zeta_0 : H \cap K \rightarrow \zeta(H) \cap \zeta(K)$ of ζ on $H \cap K$ to $\zeta(H) \cap \zeta(K)$ is surjective.*

It is worthwhile to mention that Theorem 1.1 is false in case when one of subgroups H, K has finite index in $L = \langle H, K, S(H, K) \rangle$. Indeed, if H has finite index j in L and $r(L) = n > 2$ then, according to the Schreier’s formula, we have $r(H) = (n - 1)j + 1$, see [7]. Since η is an epimorphism, the subgroup $\eta(H)$ has finite index $j' \leq j$ in $F_2 = \eta(L)$ and it follows from the Schreier’s formula that $r(\eta(H)) = j' + 1$. Hence, the equality $r(H) = r(\eta(H))$ is impossible as $(n - 1)j > j'$ and the restriction of η may not be injective on H .

We also remark that Theorem 1.1 is motivated by the author’s article [5] in which certain modifications of Stallings graphs of subgroups H, K , that do not change the ranks $r(H), r(K), r(H \cap sKs^{-1})$ for every $s \in S(H, K)$, are used to achieve some desired properties of the Stallings graph of L whose rank $r(L)$, however, might increase under carried out modifications. In this article, we make different modifications, in somewhat opposite direction, that decrease the rank $r(L)$ down to 2 while the ranks $r(H), r(K), r(H \cap sKs^{-1})$ for every $s \in S(H, K)$, and the cardinality $|S(H, K)|$ are kept fixed.

2. PRELIMINARIES

Suppose that Q is a graph. Let VQ denote the set of vertices of Q and let EQ denote the set of oriented edges of Q . If $e \in EQ$ then e^{-1} denotes the edge with the opposite to e orientation, $e^{-1} \neq e$.

For $e \in EQ$, let e_- and e_+ denote the initial and terminal, respectively, vertices of e . A path $p = e_1 \dots e_k$, where $e_i \in EQ$, $(e_i)_+ = (e_{i+1})_-$, $i = 1, \dots, k - 1$, is called *reduced* if, for every $i = 1, \dots, k - 1$, $e_i \neq e_{i+1}^{-1}$. The *length* of p is k , denoted $|p| = k$. The initial vertex of p is $p_- = (e_1)_-$ and the terminal vertex of p is $p_+ = (e_k)_+$. A path p is *closed* if $p_- = p_+$. If $p = e_1 \dots e_k$ is a closed path then a *cyclic permutation* \bar{p} of p is a path of the form $e_{1+i}e_{2+i} \dots e_{k+i}$, where $i = 1, \dots, k$ and the indices are considered $\text{mod } k$.

The subgraph of Q that consists of edges of all closed paths p of Q such that $|p| > 0$ and any cyclic permutation of p is reduced is called the *core* of X , denoted $\text{core}(X)$.

Let F be a free group of finite rank $r(F) > 1$. We consider F as the fundamental group $\pi_1(U)$ where U is a bouquet of $r(F)$ circles.

Following Stallings [10], see also [2], [6], with every (finitely generated) subgroup H of $F = \pi_1(U)$, we can associate a connected (resp. finite) graph $X = X(H)$ with a distinguished vertex $o \in VX$ and a locally injective map $\varphi : X \rightarrow U$ of graphs so that H is isomorphic to $\pi_1(X, o)$. Such a graph X of H is called a *Stallings graph* of H and the map φ is called a *canonical immersion*.

Consider two finitely generated subgroups H, K of the free group F . Pick a set $S(H, K)$ of representatives of those double cosets HgK , $g \in F$, for which the intersection $H \cap gKg^{-1}$ is nontrivial.

Let X, Y be finite Stallings graphs of the subgroups H, K , resp., and let $X \times_U Y$ denote the pullback of canonical immersions

$$\varphi_X : X \rightarrow U, \quad \varphi_Y : Y \rightarrow U. \quad (2.1)$$

According to Walter Neumann [9], the set $S(H, K)$ is finite and the nontrivial intersections $H \cap sKs^{-1}$, where $s \in S(H, K)$, are in bijective correspondence with connected components W_s of the core

$$W := \text{core}(X \times_U Y).$$

Moreover, for every $s \in S(H, K)$, we have

$$\bar{r}(H \cap sKs^{-1}) = \frac{1}{2}|EW_s| - |VW_s|,$$

where $\bar{r}(F) = \max(r(F) - 1, 0)$ is the reduced rank of a free group F and $|T|$ is the cardinality of a finite set T . Recall that, according to our notation, the number of nonoriented edges of W_s is $\frac{1}{2}|EW_s|$.

For a finite graph Q , denote

$$\bar{r}(Q) := \frac{1}{2}|EQ| - |VQ|,$$

hence, $\bar{r}(Q)$ is the negative Euler characteristic of Q .

In particular, $\bar{r}(W_s) = \bar{r}(H \cap sKs^{-1})$ and

$$\sum_{s \in S(H, K)} \bar{r}(H \cap sKs^{-1}) = \bar{r}(W) = \frac{1}{2}|EW| - |VW|.$$

Let α'_X, α'_Y denote the projection maps $X \times_U Y \rightarrow X$, $X \times_U Y \rightarrow Y$, resp. Restricting α'_X, α'_Y to $W \subseteq X \times_U Y$, we obtain immersions

$$\alpha_X : W \rightarrow X, \quad \alpha_Y : W \rightarrow Y.$$

We also consider the subgroup $L = \langle H, K, S(H, K) \rangle$ of F . Let Z denote a Stallings graph of $\langle H, K, S(H, K) \rangle$ and let $\gamma : Z \rightarrow U$ denote a canonical immersion.

Let $\beta_X : X \rightarrow Z$, $\beta_Y : Y \rightarrow Z$ be graph maps that satisfy the equalities $\varphi_X = \gamma\beta_X$, $\varphi_Y = \gamma\beta_Y$, see (2.1) and Fig. 1. Clearly, β_X and β_Y are immersions.

It follows from the definitions that, for every $Q \in \{U, W, X, Y, Z\}$, there is a canonical immersion $\varphi : Q \rightarrow U$, where $\varphi = \varphi_Q$ if $Q = X$ or $Q = Y$, $\varphi = \text{id}_U$ if $Q = U$, $\varphi = \gamma\beta_X\alpha_X = \gamma\beta_Y\alpha_Y$ if $Q = W$, and $\varphi = \gamma$ if $Q = Z$, see Fig. 1.

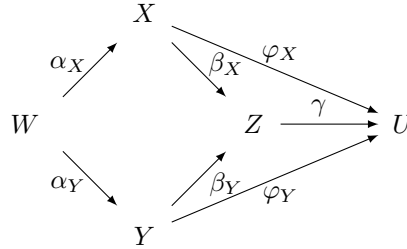


Fig. 1

It is not difficult to see that, up to a suitable conjugation of the ambient free group F , we may assume that the graphs X, Y coincide with their cores. Clearly, the same property holds for graphs U, W, Z as well.

Without loss of generality, we may also assume that $Z = U$ and $\gamma = \text{id}_Z$. For this reason, we will disregard U and γ in subsequent arguments.

3. FIVE LEMMAS

Let $A = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$ be an alphabet and let $F(A) = \langle a_1, \dots, a_n \rangle$ be a free group with free generators a_1, \dots, a_n . It will be convenient to consider elements of the free group $F(A)$ as words over the alphabet $A = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$. A letter-by-letter equality of words u, w over A is denoted $u \equiv v$. Suppose $w \equiv c_1 \dots c_\ell$ is a word over A , where $c_1, \dots, c_\ell \in A$ are letters. The length of w is denoted $|w| = \ell$. We say that a word $w \equiv c_1 \dots c_\ell$ is *reduced* if $|w| > 0$ and $c_i \neq c_{i+1}^{-1}$ for every $i = 1, \dots, \ell - 1$.

A finite graph B is called a *labeled A -graph*, or just an *A -graph*, if B is equipped with a function $\varphi : EB \rightarrow A$ so that, for every $e \in EB$, we have $\varphi(e^{-1}) = \varphi(e)^{-1}$. An A -graph B is called *irreducible* if, for every pair $e_1, e_2 \in EB$, the equalities $\varphi(e_1) = \varphi(e_2)$ and $(e_1)_- = (e_2)_-$ imply that $e_1 = e_2$. Note that an irreducible A -graph need not be connected and may contain vertices of degree < 2 .

It is easy to see that B is an irreducible A -graph with a labeling function φ if and only if there is an immersion $\varphi_0 : B \rightarrow A_0$, where A_0 is a bouquet of n oriented circles a_1, \dots, a_n so that the restriction of φ_0 on EB is φ .

We now discuss some operations over irreducible A -graphs. Let B be a finite irreducible A -graph. A connected component C of B is called *A -complete* if every vertex of C has degree $|A| = 2n$. Equivalently, the restriction of φ_0 on C , $\varphi_0|_C : C \rightarrow A_0$, is a covering. If a connected component C of B is not A -complete, we will say that C is *A -incomplete*.

Suppose $V_1 \subseteq VB$ is a subset of vertices of B , w is a reduced word over A , $w \equiv c_1 \dots c_\ell$, where $c_1, \dots, c_\ell \in A$ are letters. For every $v \in V_1$, we consider a new graph which is a path $p(v) = e_1(v) \dots e_\ell(v)$, consisting of edges $e_1(v), \dots, e_\ell(v)$ labeled by the letters c_1, \dots, c_ℓ , resp., so $\varphi(p(v)) \equiv w$. For every $v \in V_1$, we attach the path $p(v)$ to B by identifying the vertices $p(v)_-$ and v . This way we obtain a labeled A -graph $B'(V_1, w)$. We will then apply a folding process to $B'(V_1, w)$ that inductively identifies edges e and e' whenever $\varphi(e) = \varphi(e')$ and $e_- = e'_-$. As a result, we obtain an irreducible A -graph $B(V_1, w)$ that contains the original A -graph B , $B \subseteq B(V_1, w)$, and has the following properties: $\bar{r}(B(V_1, w)) = \bar{r}(B)$ and, for every $v \in V_1$, there is a unique path $p(v)$ in $B(V_1, w)$ such that $p(v)_- = v$ and $\varphi(p(v)) \equiv w$.

We also observe that $B(V_1, w) = B \cup F(V_1, w)$, where $F(V_1, w)$ is a forest, i.e., a disjoint collection of trees, and $B \cap F(V_1, w) = V_1^*$, where $V_1^* \subseteq VB$.

Lemma 3.1. *Suppose that B is a finite irreducible A -graph and no connected component of B is A -complete. Then, for every subset $V_1 \subseteq VB$, there is a reduced word $w = w(V_1)$ over A and there is an irreducible A -graph $B(V_1, w)$ that has the following properties. The graph $B(V_1, w)$ contains B as a subgraph, $B(V_1, w) = B \cup F(V_1, w)$, where $F(V_1, w)$ is a forest, $B \cap F(V_1, w) = V_1^*$, $V_1^* \subseteq VB$, and $\bar{r}(B(V_1, w)) = \bar{r}(B)$.*

Furthermore, for every vertex $v \in V_1$, there is a unique path $p(v)$ in $B(V_1, w)$ such that $p(v)_- = v$, $\varphi(p(v)) \equiv w$, $p(v)_+$ has degree one, and $p(v)_+ \notin B$.

Proof. We prove this Lemma by induction on $|V_1| \geq 1$. To make the basis step, we let $V_1 := \{v\}$. By the hypothesis of the Lemma, there is a path q in B such that $\varphi(q)$ is reduced, $q_- = v$ and $q_+ = u$, where u is a vertex of B with $\deg(u) < 2n$. Then there exists a letter $c \in A$ such that there is no edge in B with $e_- = u$ and $\varphi(e) = c$. Taking the word $w = \varphi(q)c$, we will obtain the desired result.

To make the induction step, assume that our claim holds for a set $V_1 = \{v_1, \dots, v_k\}$ with $|V_1| = k$ and $V_2 = \{v_1, \dots, v_k, v_{k+1}\}$, where $v_{k+1} \notin V_1$. By the induction hypothesis, there exists a word $w_1 = c_1 \dots c_\ell$ and an irreducible A -graph $B(V_1, w_1)$ with the properties stated in Lemma. Let m denote the maximum of distances between vertices in the same connected component of B over all components of B . We replace w_1 by the reduced word $w_2 = w_1 c_\ell^{m+2}$ and construct the graph $B(V_1, w_2)$ for this new word w_2 . It is clear that $B(V_1, w_2)$ has all of the properties of $B(V_1, w_1)$. In addition, for every path $p(v_i)$ in $B(V_1, w_2)$, there is a factorization $p(v_i) = p_1(v_i)p_2(v_i)$ so that $\varphi(p_2(v_i)) = c_\ell^{m+2}$, all the vertices of $p_2(v_i)$, except for $p_2(v_i)_+$, have degree 2 and $\deg p_2(v_i)_+ = 1$.

Now we take a new graph which is a path $p(v_{k+1})$ with $\varphi(p(v_{k+1})) = w_2$ and attach it to $B(V_1, w_2)$ so that $p(v_{k+1})_- = v_{k+1}$. Then we do foldings to produce an irreducible graph $B(V_2, w_2)$ in which there is a path that starts at v_{k+1} and is labeled by w_2 . We will denote this path by $p(v_{k+1})$. Note $B(V_1, w_2)$ is a subgraph of $B(V_2, w_2)$. Moreover, if $B(V_1, w_2) \subsetneq B(V_2, w_2)$, then $\deg p(v_{k+1})_+ = 1$ and $p(v_{k+1})_+ \notin B$, hence $B(V_2, w_2)$ has all of the desired properties and the induction step is complete. Otherwise, we may assume that $B(V_1, w_2) = B(V_2, w_2)$ and we consider two cases depending on whether or not $p(v_{k+1})_+ \in B \subseteq B(V_2, w_2)$.

First assume that $p(v_{k+1})_+ \notin B \subseteq B(V_1, w_2)$, that is, the path $p(v_{k+1})$ ends in a vertex of a tree $T \subseteq F(V_1, w_2)$, where

$$B(V_1, w_2) = B \cup F(V_1, w_2), \quad B \cap F(V_1, w_2) = V_1^*, \quad V_1^* \subseteq VB,$$

and $F(V_1, w_2)$ is a forest. Let q_T be a shortest path in T so that $(q_T)_- = p(v_{k+1})_+$ and $(q_T)_+ \notin B$, $\deg(q_T)_+ = 1$. We remark that $|q_T| > 0$ because, otherwise, $p(v_{k+1})_+ = p(v_i)_+$ for some $i, i \leq k$, whence $v_{k+1} = v_i$, contrary to $v_{k+1} \notin V_k$. Let $b \in A$ be the first letter of $\varphi(q_T)$. Since w_2 ends in c_ℓ , it follows that $b \neq (c_\ell)^{-1}$. Let $b' \in A$ be different from the letters $b, (c_\ell)^{-1}$. Then it follows from the definitions that the word

$$w_3 \equiv w_2 \varphi(q_T) b' \equiv w_1 c_\ell^{m+2} \varphi(q_T) b'$$

is reduced and can be used as a desired word w for the set $V_{k+1} = V_k \cup \{v_{k+1}\}$.

Now consider the case when $p(v_{k+1})_+ \in B \subseteq B(V_1, w_2)$. By the definition of m , the vertex $p(v_{k+1})_+$ can be joined in $B(V_1, w_2)$ with a vertex u of a tree T' , $T' \subseteq F(V_1, w_2)$, by a path q so that $|q| \leq m+1$, where $u \notin B$, and u is connected to a vertex in B by an edge of T' . Consider a reduced word w_4 equal in $F(A)$ to $w_1 c_\ell^{m+2} \varphi(q)$ which we write in the form

$$w_4 \stackrel{F(A)}{=} w_1 c_\ell^{m+2} \varphi(q).$$

Note that at most $|q|$ letters of the reduced word $w_1 c_\ell^{m+2}$ could cancel with those of $\varphi(q)$. Since $|q| \leq m+1$, the word w_4 can still be used for the set V_1 in place of w_2 . As above, we construct an irreducible A -graph $B(V_2, w_4)$ with $\varphi(p(v_i)) = w_4$ for every $i = 1, \dots, k+1$. The path $p(v_{k+1})$ with $p(v_{k+1})_- = v_{k+1}$ and $\varphi(p(v_{k+1})) = z_4$ will have its terminal vertex $p(v_{k+1})_+$ on a tree T of $F(V_1, w_4)$. Now we can argue as

in the foregoing case. We pick a shortest path q_T in T such that $(q_T)_- = p(v_{k+1})_+$, $(q_T)_+ \notin B$, $\deg(q_T)_+ = 1$, and consider a reduced word w_5 such that

$$w_5 = w_4 \varphi(q_T) c' \stackrel{F(A)}{=} w_1 c_\ell^{m+2} \varphi(q) \varphi(q_T) c',$$

where $c' \in A$ is now chosen so that c' is different from the first letter of $\varphi(q_T)$ and from b^{-1} , where b is the last letter of w_4 . As above, we see that $|q_T| > 0$. It is straightforward to verify that the graph $B(V_2, w_5)$ has all of the required properties and Lemma 3.1 is proven. \square

We now generalize Lemma 3.1 to the situation with two subsets $V_1, V_2 \subseteq VB$.

Lemma 3.2. *Suppose that B is a finite irreducible A -graph and no connected component of B is A -complete. Then, for every pair of subsets $V_1, V_2 \subseteq VB$, there are nonempty reduced words $w_1 = w_1(V_1, V_2)$, $w_2 = w_2(V_1, V_2)$ over A and there is an irreducible A -graph $B(V_1, w_1; V_2, w_2)$ such that $B(V_1, w_1; V_2, w_2)$ contains B , $B(V_1, w_1; V_2, w_2) = B \cup F(V_1, w_1; V_2, w_2)$, where $F(V_1, w_1; V_2, w_2)$ is a forest, $B \cap F(V_1, w_1; V_2, w_2) = V_{12}^*$, where $V_{12}^* \subseteq VB$, and $\bar{r}(B(V_1, w_1; V_2, w_2)) = \bar{r}(B)$.*

Furthermore, for every vertex $v_i \in V_i$, $i = 1, 2$, there is a unique path $p(v_i)$ in $B(V_1, w_1; V_2, w_2)$ such that $p(v_i)_- = v_i$, $\varphi(p(v_i)) \equiv w_i$, $p(v_i)_+$ has degree one, $p(v_i)_+ \notin B$, and the sets $\{p(v_1)_+ \mid v_1 \in V_1\}$, $\{p(v_2)_+ \mid v_2 \in V_2\}$ are disjoint.

Proof. First we apply Lemma 3.1 to the set V_1 to obtain a word $w'_1 \in F(A)$ and a graph $B(V_1, w'_1)$ with properties of Lemma 3.1. We also apply Lemma 3.1 to the set V_2 to obtain a word $w'_2 \in F(A)$ and a graph $B(V_2, w'_2)$ with properties of Lemma 3.1. Since $|A| \geq 4$, it follows that there are letters $b_1, b_2 \in A$ so that the words $w'_1 b_1$, $w'_2 b_2$ are reduced and $b_1 \notin \{b_2, (b_2)^{-1}\}$. Let $k_2 = |w'_1| + |w'_2|$. Define the words $w_1 = w'_1 b_1^{k_2}$, $w_2 = w'_2 b_2^{k_2}$. Now we attach paths $p_i(v_i)$ with $\varphi(p_i(v_i)) = w_i$, over all $v_i \in V_i$, $i = 1, 2$, to the graph B by identifying the vertices $p_i(v_i)_-$ and $v_i \in V_i \subseteq B$. Making foldings, we obtain an irreducible A -graph $B(V_1, w_1; V_2, w_2)$. It is easy to see that the graph $B(V_1, w_1; V_2, w_2)$ has all of the properties stated in Lemma 3.2. \square

As above, let B be a finite irreducible A -graph and $f \in A$. Let

$$E_f B := \{e \mid e \in EB, \varphi(e) = f\}$$

denote the set of all edges $e \in EB$ such that $\varphi(e) = f$ and, analogously, denote

$$E_{f^{-1}} B := \{e \mid e \in EB, \varphi(e) = f^{-1}\}.$$

Let B_f denote the graph $B \setminus (E_f B \vee E_{f^{-1}} B)$,

$$B_f := B \setminus (E_f B \vee E_{f^{-1}} B),$$

and, similarly, denote

$$A_f := A \setminus \{f, f^{-1}\}.$$

The restriction of the immersion $\varphi_0 : B \rightarrow A_0$ to B_f denote $\varphi_{0,f} : B_f \rightarrow A_{0,f}$, where $VA_{0,f} = \{0_f\}$ and $EA_{0,f} = A_f$. Denote

$$(E_f B)_- := \{e_- \mid e \in E_f B\}, \quad (E_f B)_+ := \{e_+ \mid e \in E_f B\}.$$

A connected component of the graph B_f is called A_f -complete if the degree of its every vertex is equal to $|A_f| = 2n - 2$. Otherwise, a connected component of the graph B_f is called A_f -incomplete.

Let B_f^{com} denote the set of all A_f -complete connected components of B_f and B_f^{inc} denote the set of all A_f -incomplete connected components of B_f . We also denote

$$V_1^{\text{inc}}(f_-) := B_f^{\text{inc}} \cap (E_f B)_-, \quad V_2^{\text{inc}}(f_+) := B_f^{\text{inc}} \cap (E_f B)_+.$$

Let us apply Lemma 3.2 to the irreducible A_f -graph B_f^{inc} and its sets of vertices $V_1^{\text{inc}}(f_-)$ and $V_2^{\text{inc}}(f_+)$. As a result, we obtain two reduced words w_1, w_2 over A_f and an irreducible A_f -graph

$$B_f^{\text{inc}}(V_1^{\text{inc}}(f_-), w_1; V_2^{\text{inc}}(f_+), w_2)$$

with the properties of Lemma 3.2.

Now we construct more labeled A -graphs in the following fashion. For every edge $e \in E_f B$, we consider a new graph which is a path $p_1(e)ep_2(e)^{-1}$ so that $\varphi(p_1(e)) \equiv w_1$, $\varphi(e) = f$, $\varphi(p_2(e)) \equiv w_2$. We replace every $e \in E_f B$ in B by such a path $p_1(e)ep_2(e)^{-1}$ and then do edge foldings to get an irreducible A -graph which we denote $B[f]$. We will call this graph $B[f]$, obtained by the described application of Lemma 3.2, an f -treatment of B . It is clear that the graph $B[f]$ can be thought of as the image of B under the automorphism of the free group $F(A)$ that takes f to $w_1fw_2^{-1}$ and takes a to a if $a \in A_f$. We also observe that the connected components of B_f and those of $B[f]_f$ are in the natural bijective correspondence. Moreover, it follows from the definitions and Lemma 3.2 that a connected component of B_f is A_f -complete if and only if its image in $B[f]_f$ is A_f -complete and, if they are both A_f -complete, then they are identical.

Suppose $c \in A, c \notin \{f, f^{-1}\}$. Using the notation analogous to what we introduced above for f , we consider the graph

$$B[f]_c := B[f] \setminus (E_c B[f] \cup E_{c^{-1}} B[f]).$$

As above, we consider a partition

$$B[f]_c = B[f]_c^{\text{inc}} \cup B[f]_c^{\text{com}},$$

where $B[f]_c^{\text{inc}}$ is the union of A_c -incomplete connected components of $B[f]_c$ and $B[f]_c^{\text{com}}$ is the union of A_c -complete connected components of $B[f]_c$.

Lemma 3.3. *Suppose B is a finite irreducible A -graph, $|A| \geq 6$, no connected component of B is a complete A -graph, $f \in A$, and $B[f]$ is an f -treatment of B . Then, for every $c \in A \setminus \{f, f^{-1}\}$, considering the sets $V(B[f]_c^{\text{com}})$, $V(B[f]_f^{\text{com}})$ as subsets of $VB[f]$, one has that*

$$V(B[f]_c^{\text{com}}) \subseteq V(B[f]_f^{\text{com}})$$

and either $V(B[f]_c^{\text{com}}) = V(B[f]_f^{\text{com}}) = \emptyset$ or $V(B[f]_c^{\text{com}}) \neq V(B[f]_f^{\text{com}})$. Moreover,

$$|V(B[f]_f^{\text{com}})| = |V(B_f^{\text{com}})|.$$

Proof. As was observed above, connected components of B_f and those of $B[f]_f$ are in the natural bijective correspondence, moreover, a connected component C of B_f is A_f -complete if and only if its image in $B[f]_f$ is A_f -complete and, if they are both A_f -complete, then they are identical. Hence, the A_f -graphs B_f^{com} and $B[f]_f^{\text{com}}$ are isomorphic and $|V(B[f]_f^{\text{com}})| = |V(B_f^{\text{com}})|$.

Now suppose $v \in V(B[f]_c^{\text{com}})$. Considering v as a vertex of $B[f]$, we see that there are edges e_1, e_2 in $B[f]$ starting at v such that $\varphi(e_1) = \varphi(e_2^{-1}) = f$. However, it follows from Lemma 3.2 that no vertex u of $B[f]$ has distinct edges e_1, e_2 starting

at v such that $\varphi(e_1) = \varphi(e_2^{-1}) = f$. This shows that $v \in V(B[f]_f^{\text{com}})$, whence, $V(B[f]_c^{\text{com}}) \subseteq V(B[f]_f^{\text{com}})$, as desired.

Finally, assume that $V(B[f]_c^{\text{com}}) = V(B[f]_f^{\text{com}}) \neq \emptyset$. Then for every vertex $v \in V(B[f]_c^{\text{com}}) \subseteq V(B[f])$ we have $\deg v = 2n - 2$ in both graphs $B[f]_f^{\text{com}}$ and $B[f]_c^{\text{com}}$. Hence, $\deg v = 2n$ in $B[f]$ and so $B[f]_c^{\text{com}}$ contains all of the vertices of $B[f]_c^{\text{com}} \neq \emptyset$. This, however, means that $B[f]$ and, hence, B contains an A -complete connected component. This contradiction to the assumption that B^{com} is empty proves that $V(B[f]_c^{\text{com}}) \subsetneq V(B[f]_f^{\text{com}})$ whenever $V(B[f]_f^{\text{com}})$ is not empty, as desired. \square

Lemma 3.4. *Suppose B is a finite irreducible A -graph, $|A| \geq 6$, and no connected component of B is a complete A -graph. Then there exists a finite sequence of $f_1, \dots, f_{\ell+1}$ -treatments of the graph B so that the resulting graph, denoted $B[f_1, \dots, f_{\ell+1}]$, has the following property. For $f \in A$, denote*

$$B[f_1, \dots, f_{\ell+1}]_f := B[f_1, \dots, f_{\ell+1}] \setminus (E_f(B[f_1, \dots, f_{\ell+1}]) \vee E_{f^{-1}}(B[f_1, \dots, f_{\ell+1}])).$$

For each $f \in A$, every connected component of $B[f_1, \dots, f_{\ell+1}]_f$ is A_f -incomplete. In addition, for every edge $e \in E(B[f_1, \dots, f_{\ell+1}])$ such that $\varphi(e) = f_{\ell+1}$, one has $\deg e_- = \deg e_+ = 2$ and, if $h_1(e)eh_2(e)$ is a reduced path in $B[f_1, \dots, f_{\ell+1}]$, where $h_1(e), h_2(e)$ are edges, then the labels $\varphi(h_1(e)), \varphi(h_2(e))$ are independent of e .

Proof. Pick an arbitrary letter $f_1 \in A$ and do an f_1 -treatment of the graph B . Assuming that $B[f_1]_{f_1}^{\text{com}} \neq \emptyset$, we can see from Lemma 3.3 that the result is a new irreducible A -graph $B[f_1]$ such that, for every $c \in A$, where $c \notin \{f_1, f_1^{-1}\}$, we have

$$V(B[f_1]_c^{\text{com}}) \subsetneq V(B[f_1]_{f_1}^{\text{com}}), \quad |V(B[f_1]_c^{\text{com}})| = |V(B[f_1]_{f_1}^{\text{com}})|.$$

Picking an edge f_2 such that $f_2 \in A$, $f_2 \notin \{f_1, f_1^{-1}\}$, we will do a second f_2 -treatment of the graph $B[f_1]$ and get a new irreducible A -graph $B[f_1, f_2] := B[f_1][f_2]$.

Assuming that $B[f_1, f_2]_{f_2}^{\text{com}} \neq \emptyset$, we obtain from Lemma 3.3 that, for every $c \in A$ such that $c \notin \{f_2, f_2^{-1}\}$, we have

$$V(B[f_1, f_2]_c^{\text{com}}) \subsetneq V(B[f_1, f_2]_{f_2}^{\text{com}}),$$

$$|V(B[f_1, f_2]_{f_2}^{\text{com}})| = |V(B[f_1]_{f_2}^{\text{com}})| < |V(B[f_1]_{f_1}^{\text{com}})| = |V(B[f_1]_{f_1}^{\text{com}})|.$$

Then we pick an edge f_3 , where $f_3 \in A$, $f_3 \notin \{f_2, f_2^{-1}\}$, and do a third f_3 -treatment of the graph $B[f_1, f_2]$ to construct an irreducible A -graph $B[f_1, f_2, f_3] = B[f_1, f_2][f_3]$ with further decreased $|V(B[f_1, f_2, f_3]_{f_3}^{\text{com}})|$ and so on. Assuming for each $j = 1, \dots, i$ that $B[f_1, \dots, f_j]_{f_j}^{\text{com}} \neq \emptyset$, we obtain from Lemma 3.3 that, for every $c \in A$ such that $c \notin \{f_j, f_j^{-1}\}$, it is true that

$$V(B[f_1, \dots, f_j]_c^{\text{com}}) \subsetneq V(B[f_1, \dots, f_j]_{f_j}^{\text{com}}),$$

$$|V(B[f_1, \dots, f_j]_{f_j}^{\text{com}})| < |V(B[f_1, \dots, f_{j-1}]_{f_{j-1}}^{\text{com}})|,$$

$$|V(B[f_1, \dots, f_j]_{f_j}^{\text{com}})| = |V(B[f_1, \dots, f_{j-1}]_{f_j}^{\text{com}})|.$$

Hence, it follows from these equalities and inequalities that

$$|V(B[f_1, \dots, f_{i+1}]_{f_{i+1}}^{\text{com}})| < \dots < |V(B[f_1, \dots, f_j]_{f_j}^{\text{com}})| < \dots$$

$$< |V(B[f_1, f_2]_{f_2}^{\text{com}})| < |V(B[f_1]_{f_1}^{\text{com}})|.$$

Therefore, the graph $B[f_1, \dots, f_{i+1}]_{f_{i+1}}^{\text{com}}$ will eventually become empty and then $B[f_1, \dots, f_{i+1}]_c^{\text{com}}$ will be empty for every $c \in A$. Thus we may suppose that $B[f_1, \dots, f_\ell]_{f_\ell}^{\text{com}}$ is empty for some $\ell \geq 1$.

We will do one more $f_{\ell+1}$ -treatment of the graph $B[f_1, \dots, f_\ell]$. Pick an edge $f_{\ell+1}$ such that $f_{\ell+1} \in A$, $f_{\ell+1} \notin \{f_\ell, f_\ell^{-1}\}$, do an $f_{\ell+1}$ -treatment of the graph $B[f_1, \dots, f_\ell]$ and obtain an irreducible A -graph $B[f_1, \dots, f_{\ell+1}] := B[f_1, \dots, f_\ell][f_{\ell+1}]$. It follows from Lemma 3.3 that, for every $c \in A$, the set $B[f_1, \dots, f_{\ell+1}]_c^{\text{com}}$ is still empty, in particular, $B[f_1, \dots, f_{\ell+1}]_f^{\text{com}} = \emptyset$. In addition, by the definitions and Lemma 3.2, we will also have the claimed property of edges $e \in EB[f_1, \dots, f_{\ell+1}]$ such that $\varphi(e) = f_{\ell+1}$. \square

Lemma 3.5. *Suppose $|A| \geq 4$, $d_1, d_2 \in A$ and w is a nonempty reduced word over A . Then there are letters $b_1, b_2 \in A$ such that the word $d_1(b_1wb_2)^2d_2$ is reduced. Furthermore, let $c \in A$, $c \notin \{b_2, b_2^{-1}\}$, and $n_w := |w| + 1$. Then the word*

$$z_4(w) := b_1wb_2c^{n_w+1}b_2b_1wb_2c^{n_w+2}b_2b_1wb_2c^{n_w+3}b_2b_1wb_2c^{n_w+4}b_2 \quad (3.1)$$

has the following properties. If $(b_2c^{n_w+i}b_2)^k$, where $k = \pm 1$, $i = 1, 2, 3, 4$, is a subword of $z_4(w)$, then $k = 1$ and the location of the subword $b_2c^{n_w+i}b_2$ is standard, i.e., it is the suffix of length $n_w + i + 2$ of the i th syllabus $b_1wb_2c^{n_w+i}b_2$ of $z_4(w)$. Moreover, the word $d_1z_4(w)d_2$ is reduced.

Proof. Since $|A| \geq 4$, there are at least two letters $x \in A$ such that the word d_1xw is reduced. Also, there are at least two letters $y \in A$ such that the word $wy d_2$ is reduced. If $d_1(xwy)^2d_2$ is not reduced then $x = y^{-1}$. For every x with d_1xw being reduced, there is at most one y with $wy d_2$ being reduced such that $d_1(xwy)^2d_2$ is not reduced. Hence there exist x and y in A such that $d_1(xwy)^2d_2$ is reduced. Denote $b_1 := x$ and $b_2 := y$.

Since $c \notin \{b_2, b_2^{-1}\}$ and the maximal power of c that can occur in b_1wb_2 is $c^{|w|+1} = c^{n_w}$, it follows from the definition (3.1) of the word $z_4(w)$ that $b_2c^{n_w+i}b_1$ can only occur in $z_4(w)^{\pm 1}$ as the suffix of length $n_w + i + 2$ of the i th syllabus $b_1wb_2c^{n_w+i}b_2$ of $z_4(w)$.

It remains to note that the word $d_1z_4(w)d_2$ is reduced because the word $d_1(xwy)^2d_2$ is reduced. \square

4. PROOF OF THEOREM 1.1

As in Sect. 2, consider the graphs W, X, Y, Z .

Lemma 4.1. *Suppose either graph Q , where $Q \in \{X, Y\}$, contains a path p_Q so that every vertex of p_Q has degree 2 in Q , $\beta_Q(p_Q) \equiv z_4(w)$, where w is a nonempty reduced word over the alphabet $A = EZ$ and $z_4(w)$ is the word defined by (3.1). Let $z_4(w) = z_{41}z_{42}$ be a factorization of $z_4(w)$ so that*

$$z_{41} \equiv b_1wb_2c^{n_w+1}b_2b_1wb_2c^{n_w+2}b_2, \quad z_{42} \equiv b_1wb_2c^{n_w+3}b_2b_1wb_2c^{n_w+4}b_2$$

and let $v_2(p_Q)$ be a vertex of the path p_Q that defines a corresponding factorization $p_Q = p_{Q1}p_{Q2}$ so that $\varphi(p_{Q1}) \equiv z_{41}$, $\varphi(p_{Q2}) \equiv z_{42}$.

Furthermore, assume that there exists a vertex $u \in VW$, where $W = \text{core}(X \times_Z Y)$, such that either $\alpha_X(u) = v_2(p_X)$ and $\alpha_Y(u) \in p_Y$ or $\alpha_X(u) \in p_X$ and $\alpha_Y(u) = v_2(p_Y)$. Then $\alpha_X(u) = v_2(p_X)$, $\alpha_Y(u) = v_2(p_Y)$, and there is a path p in W such that $\alpha_X(p) = p_X$ and $\alpha_Y(p) = p_Y$.

Proof. For definiteness, suppose $u \in VW$ is such that $\alpha_X(u) = v_2(p_X)$ and $\alpha_Y(u) \in p_Y$. Let $p_Y = p_{Y1}p_{Y2}$ be the factorization of p_Y defined by the vertex $\alpha_Y(u)$. Pick a path p_{Yi^*} , $i^* = 1, 2$, such that $|p_{Yi^*}| \geq \frac{1}{2}|p_Y|$. Since the factorization of p_X defined by the vertex $\alpha_X(u) = v_2(p_X)$ defines the factorization $z_4(w) = z_{41}z_{42}$ of the word $z_4(w) = \beta_X(p_X)$, it follows that if $i^* = 2$ then the word $\beta_Y(p_{Y2})$ contains a subword $b_2c^{n_w+3}b_2$ or contains a subword $b_2^{-1}c^{-n_w-2}b_2^{-1}$. The second subcase, however, is impossible by Lemma 3.5. Similarly, if $i^* = 1$ then the word $\gamma\beta_Y(p_{Y1})$ contains a subword $b_2c^{n_w+2}b_2$ or contains a subword $b_2^{-1}c^{-n_w-3}b_2^{-1}$. The second subcase is impossible by Lemma 3.5. In either case $i^* = 1, 2$, we can apply Lemma 3.5 to the subword $b_2c^{n_w+3}b_2$ or the subword $b_2c^{n_w+2}b_2$ of $\beta_Y(p_Y)$ and obtain the desired conclusion. \square

We now define a special type of transformations over the graphs W, X, Y, Z , called (f, p) -transformations.

Suppose that f is an edge of the graph Z and $p = e_1 \dots e_\ell$ is a path in Z such that $p_- = (e_1)_- = f_-$, $p_+ = (e_\ell)_+ = f_+$, and there are no occurrences of f , f^{-1} among edges e_1, \dots, e_ℓ .

Let $Q \in \{W, X, Y, Z\}$ and $\varphi : Q \rightarrow Z = U$ be a canonical immersion. Consider a set H_f of all edges h in $EW \vee EX \vee EY \vee EZ$ that are sent by φ to f . We replace every edge $h \in H_f$ with a path $p(h) = e_1(h) \dots e_\ell(h)$ such that

$$\varphi(e_i(h)) := \varphi(e_i)$$

for every $i = 1, \dots, \ell$, and we extend every map $\nu \in \{\alpha_X, \dots, \beta_Y\}$ to paths $p(h)$ so that if $\nu : Q \rightarrow Q'$, where $Q, Q' \in \{W, X, Y, Z\}$, and $\nu(h) = h'$, then we extend ν to paths $p(h)$, $p(h')$ by setting $\nu(e_1(h)) := e_1(h')$, \dots , $\nu(e_\ell(h)) := e_\ell(h')$. Thus obtained graphs we denote by Q_{fp} , where $Q \in \{W, X, Y, Z\}$, and thus obtained maps we denote by $\alpha_{X_{fp}}, \dots, \beta_{Y_{fp}}$.

We now perform folding process over every modified graph Q_{fp} , $Q \in \{W, X, Y, Z\}$. Recall that this is an inductive procedure which identifies every pair of oriented edges $g_1, g_2 \in EQ_{fp}$ whenever $(g_1)_- = (g_2)_-$ and g_1, g_2 have the same labels $\varphi(g_1)$, $\varphi(g_2)$.

Note that folding process decreases by one the reduced rank $r(Z)$ because the new path $p(f) = e_1(f) \dots e_\ell(f)$ that replaces the edge f in Z will be attached to the path $p = e_1 \dots e_\ell$ in $Z \setminus \{f, f^{-1}\}$ thus producing a graph \bar{Z}_{fp} with $r(\bar{Z}_{fp}) = r(Z) - 1$. Clearly, \bar{Z}_{fp} is a bouquet of $r(Z) - 1$ circles and $\text{core}(\bar{Z}_{fp}) = \bar{Z}_{fp}$.

Furthermore, when a folding process applied to Q_{fp} , $Q \in \{W, X, Y, Z\}$, is complete and produces a graph \bar{Q}_{fp} , we will take the core of \bar{Q}_{fp} thus obtaining a graph $Q_p = \text{core}(\bar{Q}_{fp})$. It follows from the definitions that we will have maps $\alpha_{X_p}, \dots, \beta_{Y_p}$ with properties of original maps α_X, \dots, β_Y . This alteration of the graphs W, X, Y, Z by means of an edge $f \in EZ$ and a path p in Z will be called an (f, p) -transformation over the graphs W, X, Y, Z .

We will say that an (f, p) -transformation is *conservative* if it preserves the numbers $\bar{r}(X)$, $\bar{r}(Y)$, $\bar{r}(W_s)$ for every connected component W_s of W , $s \in S(H, K)$, and the core $\text{core}(X_p \times_{Z_p} Y_p)$ of the pullback $X_p \times_{Z_p} Y_p$ coincides with the graph W_p . Thus a conservative (f, p) -transformation decreases $\bar{r}(Z)$ by one while keeping the numbers $\bar{r}(X)$, $\bar{r}(Y)$, $\bar{r}(W)$ unchanged.

Here is our principal technical result that will be used to prove Theorem 1.1.

Lemma 4.2. *Suppose $\bar{r}(Z) \geq 2$, $U = Z$, $\gamma = \text{id}_Z$, and neither of the maps $\beta_X : X \rightarrow Z$, $\beta_Y : Y \rightarrow Z$ is a covering. Then there is an automorphism τ of the*

free group $F(EZ) = \pi_1(Z)$ such that Stallings graphs of subgroups $\tau(H)$, $\tau(K)$, $\tau(\bigvee_{s \in S(H,K)} (H \cap sKs^{-1}))$, denoted X^τ , Y^τ , W^τ , resp., have no vertices of degree 1 and there exists a conservative (f, p) -transformation over the graphs W^τ , X^τ , Y^τ , $Z^\tau = Z$.

Proof. Assume that neither of the maps $\beta_X : X \rightarrow Z$, $\beta_Y : Y \rightarrow Z$ is a covering. Consider the disjoint union $X \vee Y$ of X, Y as a graph B and the set EZ as the alphabet A . Let $\varphi : EB \rightarrow A$ be defined on EX as the restriction of β_X and on EY as the restriction of β_Y . We also consider W as an A -graph by using the map $\varphi : EW \rightarrow A$, where φ is the restriction of $\beta_X \alpha_X = \beta_Y \alpha_Y$.

Since neither of X, Y is A -complete, we can apply Lemma 3.4 and find a sequence of f_1 -, \dots , $f_{\ell+1}$ -treatments for B which transform the graph $B = X \vee Y$ into $B[f_1, \dots, f_{\ell+1}]$ with the properties of Lemma 3.4. Note that an f -treatment of $B = X \vee Y$ can equivalently be described as an application of a suitable automorphism τ_f of the free group $F(A) = \pi_1(Z)$. We specify that this automorphism τ_f is the composition of an automorphism θ_f , given by $\theta_f(f) = w_1 f w_2^{-1}$, $\theta_f(a) = a$ for every $a \in A_f$, where w_1, w_2 are some words over A_f , and an inner automorphism of $F(A)$ applied, if necessary, to move the base vertices of graphs $\theta_f(X)$, $\theta_f(Y)$, representing subgroups $\theta_f(H)$, $\theta_f(K)$, resp., so that the base vertices, after the move, would be in $\text{core}(\theta_f(H))$, $\text{core}(\theta_f(K))$ and the correspondence between graphs and subgroups would be preserved. Therefore, the composition of these f_1 -, \dots , $f_{\ell+1}$ -treatments can be induced by a suitable automorphism of the free group $F(A)$ applied to subgroups H, K , $\bigvee_{s \in S(H,K)} (H \cap sKs^{-1})$ and to corresponding Stallings graphs X, Y, W . By Lemmas 3.4, 3.2, we may assume that, for every $c \in A$, the graph $B[f_1, \dots, f_{\ell+1}]_c$ has no A_c -complete component and that every edge $e \in EB[f_1, \dots, f_{\ell+1}]$ such that $\varphi(e) = f_{\ell+1}$ is contained in a reduced path $h_1(e)eh_2(e)$, where $h_1(e), h_2(e)$ are edges, so that $\deg e_- = \deg e_+ = 2$ and the letters $\varphi(h_1(e))$, $\varphi(h_2(e))$ are independent of e .

Denote

$$\varphi(h_1(e)) = d_1, \quad \varphi(h_2(e)) = d_2, \quad (4.1)$$

where $d_1, d_2 \in A_{f_{\ell+1}} = EZ \setminus \{f_{\ell+1}, f_{\ell+1}^{-1}\}$. Since $B = X \vee Y$, we can represent the graph $B[f_1, \dots, f_{\ell+1}]$ in the form

$$B[f_1, \dots, f_{\ell+1}] = X[f_1, \dots, f_{\ell+1}] \vee Y[f_1, \dots, f_{\ell+1}],$$

where $X[f_1, \dots, f_{\ell+1}] = X^\tau$ is obtained from X by these f_1 -, \dots , $f_{\ell+1}$ -treatments and $Y[f_1, \dots, f_{\ell+1}] = Y^\tau$ is obtained from Y by the f_1 -, \dots , $f_{\ell+1}$ -treatments. Similarly, let $W[f_1, \dots, f_{\ell+1}] = W^\tau$ denote the graph obtained from W by the f_1 -, \dots , $f_{\ell+1}$ -treatments.

Consider the irreducible $A_{f_{\ell+1}}$ -graph

$$B[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}} = X[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}} \vee Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$$

and apply Lemma 3.1 to this graph and to the vertex set $V_1 = VB[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$. According to Lemma 3.1, there exists a reduced nonempty word w over the alphabet $A_{f_{\ell+1}} = EZ \setminus \{f_{\ell+1}, f_{\ell+1}^{-1}\}$ with the properties of Lemma 3.1. In particular, for every vertex $v \in VB[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$, the $A_{f_{\ell+1}}$ -graph $B[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$ contains no path p such that $p_- = v$ and $\varphi(p) \equiv w$.

Since $A = EZ$ and $|A| \geq 6$, we have $|A_{f_{\ell+1}}| \geq 4$. Hence, Lemma 3.5 applies to the word w , to the alphabet $A_{f_{\ell+1}}$ and to the letters $d_1, d_2 \in A_{f_{\ell+1}}$ defined

by equalities (4.1). By Lemma 3.5, there are letters $b_1, b_2 \in A_{f_{\ell+1}}$ such that the word $d_1(b_1wb_2)^2d_2$ is reduced. Furthermore, let $c \in A_{f_{\ell+1}}$, $c \notin \{b_2, b_2^{-1}\}$, and $n_w = |w| + 1$. Then the word $z_4(w)$ over $A_{f_{\ell+1}}$, given by formula (3.1), has the properties stated in Lemma 3.5.

Now we perform an $(f_{\ell+1}, z_4(w))$ -transformation over the graphs

$$W[f_1, \dots, f_{\ell+1}] = W^\tau, \quad X[f_1, \dots, f_{\ell+1}] = X^\tau, \quad Y[f_1, \dots, f_{\ell+1}] = Y^\tau, \quad Z.$$

Making this transformation turns every edge e such that $\varphi(e) = f_{\ell+1}$ into a path $p = p(e)$ such that $p_- = e_-$, $p_+ = e_+$, and $\varphi(p) \equiv z_4(w)$. Let $B(f_{\ell+1}, z_4)$, $X(f_{\ell+1}, z_4)$, $Y(f_{\ell+1}, z_4)$, $W(f_{\ell+1}, z_4)$ denote the resulting $A_{f_{\ell+1}}$ -graphs.

Observe that if $\mu : F(A) \rightarrow F(A_{f_{\ell+1}})$ is the epimorphism defined by $\mu(b) = b$ for every $b \in A_{f_{\ell+1}}$ and $\mu(f_{\ell+1}) = z_4(w)$ then we have

$$\begin{aligned} W(f_{\ell+1}, z_4) &= \mu(W[f_1, \dots, f_{\ell+1}]) = \mu(W^\tau), \\ X(f_{\ell+1}, z_4) &= \mu(X[f_1, \dots, f_{\ell+1}]) = \mu(X^\tau), \\ Y(f_{\ell+1}, z_4) &= \mu(Y[f_1, \dots, f_{\ell+1}]) = \mu(Y^\tau). \end{aligned}$$

A path $p = p(e)$ in graphs $X(f_{\ell+1}, z_4)$, $Y(f_{\ell+1}, z_4)$, $W(f_{\ell+1}, z_4)$ such that $\varphi(p) \equiv z_4(w)$ and p results from an edge e with $\varphi(e) = f_{\ell+1}$ will be called a *standard z_4 -path*. Let p be a standard z_4 -path and $p = p_1p_2p_3p_4$ be a factorization of p so that $\varphi(p_i) \equiv b_1wb_2c^{n+i}b_2$, where $i = 1, 2, 3, 4$. Denote $v_2(p) := (p_2)_+$.

Recall that Z has a single vertex and an $(f_{\ell+1}, z_4(w))$ -transformation converts Z into $Z_{f_{\ell+1}} = Z \setminus \{f_{\ell+1}, f_{\ell+1}^{-1}\}$ such that $EZ_{f_{\ell+1}} = A_{f_{\ell+1}}$. Let

$$\varphi_{f_{\ell+1}} : B(f_{\ell+1}, z_4) = X(f_{\ell+1}, z_4) \vee Y(f_{\ell+1}, z_4) \rightarrow Z_{f_{\ell+1}}$$

denote the corresponding immersion. Consider the pullback

$$X(f_{\ell+1}, z_4) \times_{Z_{f_{\ell+1}}} Y(f_{\ell+1}, z_4)$$

and its core $\widetilde{W} := \text{core}(X(f_{\ell+1}, z_4) \times_{Z_{f_{\ell+1}}} Y(f_{\ell+1}, z_4))$.

Let us prove the equality $\widetilde{W} = W(f_{\ell+1}, z_4)$.

Let

$$\widetilde{\alpha}_X : \widetilde{W} \rightarrow X(f_{\ell+1}, z_4), \quad \widetilde{\alpha}_Y : \widetilde{W} \rightarrow Y(f_{\ell+1}, z_4)$$

denote the projection maps. Suppose $u \in V\widetilde{W}$ is a vertex such that $\widetilde{\alpha}_X(u) = v_2(p_X)$, where p_X is a standard z_4 -path of $X(f_{\ell+1}, z_4)$.

First we assume that

$$\widetilde{\alpha}_Y(u) \in Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}, \quad (4.2)$$

here the graph $Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$ is regarded as a subgraph of $Y(f_{\ell+1}, z_4)$.

Note that the conclusion of Lemma 3.1 holds for the graph $B[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$, for the set $V_1 = VB[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$ and for the reduced word $w_1 \equiv b_1w$ in place of w . Hence, the graph $Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}}$ contains no path r such that r starts at $\widetilde{\alpha}_Y(u)$ and r has the label $\varphi_{f_{\ell+1}}(r) \equiv b_1w$. Since there is a path p' in $X(f_{\ell+1}, z_4)$ such that p' starts at $\widetilde{\alpha}_X(u) = v_2(p_X)$, $\varphi_{f_{\ell+1}}(p') \equiv b_1wb_2c^{n+3}b_2$, and all vertices of p' have degree 2, it follows that there is a path q in $Y(f_{\ell+1}, z_4)$ such that q starts at $\widetilde{\alpha}_Y(u)$ and

$$\varphi_{f_{\ell+1}}(q) \equiv \varphi_{f_{\ell+1}}(p') \equiv b_1wb_2c^{n_w+3}b_2.$$

Let $q = q_1q_2$ be the factorization of q defined so that

$$\varphi_{f_{\ell+1}}(q_1) \equiv b_1w, \quad \varphi_{f_{\ell+1}}(q_2) \equiv b_2c^{n_w+3}b_2.$$

By the above observation based on Lemma 3.1, the path q_1 may not be entirely contained in $Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}} \subseteq Y(f_{\ell+1}, z_4)$. On the other hand, if $(q_1)_+ \in Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}} \subseteq Y(f_{\ell+1}, z_4)$, then, in view of (4.2), the path q_1 would have to contain a standard z_4 -path of $Y(f_{\ell+1}, z_4)$ which is impossible for $|q_1| < |z_4(w)|$. Therefore, $(q_1)_+ \notin Y[f_1, \dots, f_{\ell+1}]_{f_{\ell+1}} \subseteq Y(f_{\ell+1}, z_4)$ and the vertex $(q_1)_+$ must belong to a standard z_4 -path p_Y of $Y(f_{\ell+1}, z_4)$.

Since $|q_1| = |w| + 1$, $|q_2| = |w| + 6$, and $|p_Y| = |z_4(w)| = 8|w| + 26$, it follows from (4.2) and from $(q_1)_+ \in p_Y$ that q_2 is a subpath of $p_Y^{\pm 1}$ and there is a factorization $p_Y = p_{Y1}p_{Y2}$ defined by the vertex $(q_2)_+ \in p_Y$, where a shortest path out of p_{Y1}, p_{Y2} contains $q_2^{\pm 1}$ and

$$\min(|p_{Y1}|, |p_{Y2}|) \leq |q| = 2|w| + 7. \quad (4.3)$$

On the other hand, if $p_Y = p_{Y3}p_{Y4} = p_{Y5}p_{Y6}$ are factorizations of p_Y so that the words $\varphi_{f_{\ell+1}}(p_{Y3})$, $\varphi_{f_{\ell+1}}(p_{Y6})$ contain the standard occurrence of the word $b_2c^{n_w+3}b_2$ in $z_4(w)$, then $|p_{Y3}| \geq 6|w| + 18$ and $|p_{Y6}| \geq 3|w| + 14$. These inequalities, in view of (4.3), mean that the occurrence of

$$(b_2c^{n_w+3}b_2)^{\pm 1} = \varphi_{f_{\ell+1}}(q_2)^{\pm 1}$$

in $z_4(w) \equiv \varphi_{f_{\ell+1}}(p_Y)$ is not standard. This contradiction to Lemma 3.5 proves that the inclusion (4.2) is impossible.

Thus it is shown that, for every vertex $u \in V\widetilde{W}$, if $\tilde{\alpha}_X(u) = v_2(p_X)$ for some standard z_4 -path p_X of $X(f_{\ell+1}, z_4)$, then $\tilde{\alpha}_Y(u) \in p_Y$, where p_Y is a standard z_4 -path in $Y(f_{\ell+1}, z_4)$. Switching the graphs $X(f_{\ell+1}, z_4)$ and $Y(f_{\ell+1}, z_4)$ in the above arguments, we can analogously show that, for every vertex $u \in \widetilde{W}$, if $\tilde{\alpha}_Y(u) = v_2(p_Y)$ for some standard z_4 -path p_Y of $Y(f_{\ell+1}, z_4)$, then $\tilde{\alpha}_X(u) \in p_X$, where p_X is a standard z_4 -path in $X(f_{\ell+1}, z_4)$.

Now we can use Lemma 4.1 to conclude that, for every vertex $u \in \widetilde{W}$, if $\tilde{\alpha}_X(u) = v_2(p_X)$, where p_X is a standard z_4 -path of $X(f_{\ell+1}, z_4)$, or if $\tilde{\alpha}_Y(u) = v_2(p_Y)$, where p_Y is a standard z_4 -path of $Y(f_{\ell+1}, z_4)$, then, in either case, $\tilde{\alpha}_X(u) = v_2(p_X)$ and $\tilde{\alpha}_Y(u) = v_2(p_Y)$, where both p_X, p_Y are standard z_4 -paths. In addition, there exists a path p_W in \widetilde{W} such that $\tilde{\alpha}_X(p_W) = p_X$ and $\tilde{\alpha}_Y(p_W) = p_Y$. Consequently, if p is a path in \widetilde{W} such that one of $\tilde{\alpha}_X(p), \tilde{\alpha}_Y(p)$ is a standard z_4 -path, then both $\tilde{\alpha}_X(p), \tilde{\alpha}_Y(p)$ must be standard z_4 -paths. Now the desired equality $\widetilde{W} = W(f_{\ell+1}, z_4)$ becomes apparent.

Since $\bar{r}(X) = \bar{r}(X(f_{\ell+1}, z_4)) = \bar{r}(\mu(X^\tau))$ and $\bar{r}(Y) = \bar{r}(Y(f_{\ell+1}, z_4)) = \bar{r}(\mu(Y^\tau))$, this $(f_{\ell+1}, z_4(w))$ -transformation over the graphs $W^\tau, X^\tau, Y^\tau, Z$ is conservative, as required. It remains to note that $A_{f_{\ell+1}}$ -graphs

$$X(f_{\ell+1}, z_4) = \mu(X^\tau), \quad Y(f_{\ell+1}, z_4) = \mu(Y^\tau)$$

are $A_{f_{\ell+1}}$ -incomplete and Lemma 4.2 is proven. \square

Proof of Theorem 1.1. Since neither of subgroups H, K has finite index in $L = \langle H, K, S(H, K) \rangle$, we conclude that $r(L) \geq 2$. There is nothing to prove if $r(L) = 2$. Hence, we may assume that $r(L) = n > 2$. It follows from Lemma 4.2 and the definitions that there is an epimorphism $\eta_1 : L \rightarrow F_{n-1}$, where F_{n-1} is a free group of rank $n - 1$, with the following properties. The restriction of η_1 on H and on K is injective, a set $S(\eta_1(H), \eta_1(K))$ for subgroups $\eta_1(H), \eta_1(K)$ of F_{n-1} can be taken

to be $\eta_1(S(H, K))$, for every $s \in S(H, K)$, the restriction

$$\eta_{1,s} : H \cap sKs^{-1} \rightarrow \eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$$

of η_1 on $H \cap sKs^{-1}$ to $\eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$ is surjective, and neither of $\eta_1(H), \eta_1(K)$ has finite index in the group

$$F_{n-1} = \eta_1(L) = \langle \eta_1(H), \eta_1(K), \eta_1(S(H, K)) \rangle.$$

Iterating this argument, we obtain a desired epimorphism $\eta : L \rightarrow F_2$. \square

Proof of Corollary 1.2. If one of subgroups H, K has infinite index in their join $\langle H, K \rangle$ then one of H, K has also infinite index in the generalized join $L = \langle H, K, S(H, K) \rangle$ and Theorem 1.1 applies. By Theorem 1.1, there is an epimorphism $\eta : L \rightarrow F_2$ that has the required properties of ζ . \square

REFERENCES

- [1] D. F. Cummins and S. V. Ivanov, *Embedding of groups and quadratic equations over groups*, preprint, [arXiv:1607.06784](#) [[math.GR](#)].
- [2] W. Dicks, *Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture*, *Invent. Math.* **117**(1994), 373–389.
- [3] V. S. Guba, *Finitely generated complete groups*, *Izv. Akad. Nauk SSSR Ser. Mat.* **50**(1986), 883–924.
- [4] G. Higman, B. H. Neumann, and H. Neumann, *Embedding theorems for groups*, *J. London Math. Soc.* **24**(1949), 247–254.
- [5] S. V. Ivanov, *On joins and intersections of subgroups in free groups*, preprint, [arXiv:1607.04890](#) [[math.GR](#)].
- [6] I. Kapovich and A. G. Myasnikov, *Stallings foldings and subgroups of free groups*, *J. Algebra* **248**(2002), 608–668.
- [7] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
- [8] H. Neumann, *On the intersection of finitely generated free groups*, *Publ. Math.* **4**(1956), 186–189; Addendum, *Publ. Math.* **5**(1957), 128.
- [9] W. D. Neumann, *On the intersection of finitely generated subgroups of free groups*, *Lecture Notes in Math. (Groups-Canberra 1989)* **1456**(1990), 161–170.
- [10] J. R. Stallings, *Topology of finite graphs*, *Invent. Math.* **71**(1983), 551–565.
- [11] A. Yu. Ol’shanskii, *Geometry of defining relations in groups*, Nauka, Moscow, 1989; English translation: *Math. and Its Applications, Soviet series*, vol. 70, Kluwer Acad. Publ., 1991.

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